Algebraic Geometry Lecture 27 – Complex Multiplication of Abelian Varieties

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§1 Abelian Varieties over $\mathbb{C}$

An abelian variety is a projective variety with a group structure.

Recall, an elliptic curve E over $\mathbb{C}$ is isomorphic to a complex torus $\mathbb{C}/\Lambda$ for some lattice Λ. Something similar is true for abelian varieties, i.e. $A(\mathbb{C}) \cong \mathbb{C}^d/\Lambda$ for some $d = \dim A$ and some full lattice Λ in $\mathbb{C}^d$.

We will study the endomorphism ring $\mathrm{End}(A)$, but a more natural object to consider is $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$, which makes it into a $\mathbb{Q}$-algebra.

§2 CM-Fields and Complex Multiplication

Definition. An algebraic number field E is a CM-field if it is a totally imaginary quadratic extension of a totally real field.

Example. The cyclotomic field $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive nth root of unity is a CM-field. It is a quadratic imaginary extension of $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$.

Definition. An abelian variety (over $\mathbb{C}$), A, is said to have complex multiplication by a CM-field E if:

- $E \subset \mathrm{End}^0(A)$,
- $[E : \mathbb{Q}] = 2 \dim A$.

We'll show how to construct all abelian varieties that have complex multiplication by a given CM-field.

Let $[E : \mathbb{Q}] = 2d$. We can do this since E is a quadratic extension of something. The embeddings $E \hookrightarrow \mathbb{C}$ fall into complex conjugate pairs $(\phi, \overline{\phi})$. Define a CM-type to be a choice of d embeddings, no two of which differ by complex conjugation. Write $\Phi = \{\phi_1, \ldots, \phi_d\}$ for a CM-type. Let Φ also denote the map $\Phi : E \to \mathbb{C}^d$ given by

$$\Phi : x \mapsto (\phi_1(x), \ldots, \phi_d(x)).$$

Define $A = \mathbb{C}^d/\Phi(\mathcal{O}_E)$. This is a complex torus, hence an abelian variety. It has CM by E, since any $x \in \mathcal{O}_E$ gives rise to an endomorphism $\Phi(x)$ on A.

§3 Abelian Varieties over Finite Fields

Abelian varieties over $\mathbb{F}_q$ are important in the study of zeta functions. An abelian variety over $\mathbb{F}_q$ has a Frobenius endomorphism π_A, which commutes with all other endomorphisms, so it lies in the centre of $\mathrm{End}^0(A)$. In fact, if A is simple then $\mathrm{End}^0(A)$ is a division algebra, so $\mathbb{Q}(\pi_A)$ is a field.

Definition. A Weil q-integer is an algebraic integer π such that $|\pi| = q^{1/2}$. We say two Weil q-integers are conjugate and write $\pi \sim \pi'$ if and only if one of the following equivalent conditions holds:

- π and π' have the same minimal polynomial over $\mathbb{Q}$;
- there exists an isomorphism $\mathbb{Q}(\pi) \cong \mathbb{Q}(\pi')$;
- π and π' lie in the same orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

We denote the set of all Weil q-integers as $W(q)$.

Theorem (Honda–Tate)**.** *The map taking $A \to \pi_A$ defines a bijection between the sets*

$$\{\textit{simple abelian varieties over } \mathbb{F}_q \textit{ up to isogeny}\} \quad \longleftrightarrow \quad W(q)/\sim .$$